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A Theorem on Families of Sets

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We prove the following result and transfinite extensions of it: Let $(M_i: i \in I)$ be a family of non-zero subsets of the set S. If the cardinalities $|I| = f$ and $|S| = n$ are finite and $f > n(r-1)$, then one can find r disjoint subsets I_v $(v = 1,..., r)$ of I for which

$$\bigcup_{i \in I_1} M_i = \cdots = \bigcup_{i \in I_r} M_i .$$

The proof is constructive. We apply a generalization by R. Rado of P. Hall's celebrated theorem on systems of representatives.

Another proof of the above result has been found by H. Tverberg (see [3]). Tverberg applies his generalization of Radon's theorem (see [2]). He also shows by an example that the result is in a sense best possible.

The result we shall prove is divided in two parts, A and B. In the proof of A we shall apply a theorem of R. Rado quoted below. Part B is proved with the aid of cardinal arithmetic and we rely there on the axiom of choice.

THEOREM. *Let $\mathscr{F} = (M_i : i \in I)$ be a family of non-zero subsets of a set S. $M_i = M_j$ for $i \neq j$ is permitted.*

(A) *If the cardinalities $|I| = f$ and $|S| = n$ are finite and $f > n(r-1)$ for an integer r, then it is possible to find r non-empty disjoint subsets $I_v(v = 1,..., r)$ of I such that*

$$\bigcup_{i \in I_1} M_i = \cdots = \bigcup_{i \in I_r} M_i . \tag{1}$$

(B) *If I is infinite and $|S| < |I|$ then there are, for every cardinal*

$\mathfrak{r} < |I|$, *$\mathfrak{r}$ non-zero, disjoint subsets I_v, $v \in \omega$, (ω is an ordinal with $|\omega| = \mathfrak{r}$) of I, such that*

$$\bigcup_{i \in I_1} M_i = \cdots = \bigcup_{i \in I_\omega} M_i . \tag{2}$$

It we put $r = 2$ in (A), it is easy to find a proof using the linear dependence of columns in the associated incidence matrix. By the method in this paper one can give another proof with the aid of P. Hall's theorem on systems of distinct representatives. We shall need a generalization by R. Rado of Hall's theorem [1, p. 530].

RADO'S THEOREM. Let $(M_i : i \in I)$ be a family of subsets of E and assume that if I is infinite, then all M_i are finite. Furthermore, let r be a natural number. Then the family $(M_i : i \in I)$ possesses a system of representatives in which no element of E occurs more than r timee if and only if, for each natural number $k \leqslant |I|$, the union of any k M_i's contains at least k/r elements.

Proof of the theorem. (A) The proof will be by induction on the number of elements in S, when S is finite. For $n = 1$ the result is trivial. Assume that (A) is true, when $|S| < n$.

Let $\mathscr{F}' = (M_i : i \in I')$, $I' \subseteq I$, be a subfamily of $(M_i : i \in I)$ where $|I'| = k \leqslant n(r-1)$. Assume that the union of the sets in $\mathscr{F}'$ contains e elements, $e < k/(r-1)$. Since $e < n$ and $k > e(r-1)$, (1) follows by assumption for $\mathscr{F}'$ and then for $\mathscr{F}$.

We shall then assume that there is no subfamily $\mathscr{F}'$ of the above kind, i.e., we assume that the union of any subfamily of $k \leqslant n(r-1)$ sets from $\mathscr{F}$ contains at least $k/(r-1)$ elements. Let $\mathscr{G} = (M_i : i \in J)$ be a subfamily of $\mathscr{F}$ with $|J| = n(r-1)$. By Rado's theorem $\mathscr{G}$ possesses a system of representatives in which no element occurs more than $r-1$ times. Since $|J| = n(r-1)$, it follows that each element of S occurs $r-1$ times in this system of representatives.

To simplify notations, let $S = \{1, 2,..., n\}$, $I = \{1,..., n(r-1)+1\}$, and $J = \{1, 2,..., n(r-1)\}$. We shall write $M[i]$ instead of M_i. We change indices for the sets such that

$$i \in \bigcap_{j=1}^{r-1} M[(i-1)(r-1)+j], \qquad \text{for} \quad i \in S. \tag{3}$$

Let $\varnothing$ denote the empty set. We define the subsets of S $S_{j,v}$ ($j = 1,..., r$; $v = 0, 1,...$) recursively by

$$S_{1,0} = M[n(r-1)+1], \tag{4}$$
$$S_{2,0} = \cdots = S_{r,0} = \varnothing, \tag{5}$$
$$S_v = \bigcup_{j=1}^{r} S_{j,v}, \tag{6}$$
$$S_{j,v+1} = S_{j,v} \cup \bigcup_{i \in S_v - S_{j,v}} M[(i-1)(r-1)+\alpha_i(j)], \tag{7}$$

where $j \to \alpha_i(j)$ for fixed i is a 1-1 mapping from the set $\{j: i \in S_v - S_{j,v}\}$, which contains at most $r - 1$ elements, into the set $\{1, 2, \ldots, r - 1\}$. From (3), (6) and (7) it follows that

$$S_v \subseteq S_{j,v+1} \subseteq S_{v+1}\,, \text{ for } v = 0, 1, \ldots\,. \tag{8}$$

Since S is finite we conclude that there exists a $v = p$ such that

$$S_p = S_{1,p} = S_{2,p} = \cdots = S_{r,p}\,. \tag{9}$$

Observe that $(i - 1)(r - 1) + \alpha_i(j) = (i' - 1)(r - 1) + \alpha_{i'}(j')$ implies first $i = i'$, since $1 \leqslant \alpha_i(j) \leqslant r - 1$, and then $j = j'$, since the mapping $j \to \alpha_i(j)$ is 1-1. It follows now from (7) that each $S_{j,p}$ is the union of $M[i]$'s with different i's and no $M[i]$ is in the union for two different j's. We have proved (1) when I is finite.

(B) It suffices to prove (2) for $\mathfrak{r}$ such that $\mathfrak{n} \leqslant \mathfrak{r} < \mathfrak{f}$, where $\mathfrak{n} = |S|$ and $\mathfrak{f} = |I|$. We define

$$I_x = \{i: x \in M_i\}, \qquad \text{for } x \in S, \tag{10}$$

$$A = \{x: |I_x| \leqslant \mathfrak{r}\}, \tag{11}$$

$$J = \bigcup_{x \in A} I_x\,, \tag{12}$$

$$M = \bigcup_{i \in I - J} M_i\,. \tag{13}$$

Observe that

$$|J| \leqslant \mathfrak{n}\mathfrak{r} = \mathfrak{r}. \tag{14}$$

Since $|J| < |I|$ and $J \subseteq I$, it follows that the set M defined in (13) is non-zero. M and A are disjoint sets, for if $x \in M_i \cap A$ then $i \in I_x \subseteq J$. Hence

$$|I_x| > \mathfrak{r} \qquad \text{for} \quad x \in M. \tag{15}$$

It follows from (14) and (15) that

$$|I_x - J| > \mathfrak{r} \qquad \text{for} \quad x \in M. \tag{16}$$

We shall now define I_x' for $x \in M$ such that $I_x' \subseteq I_x - J$, $|I_x'| = \mathfrak{r}$ and $I_x' \cap I_y' = \varnothing$ for $x \neq y$. This is done by transfinite induction. Let the relation W well-order M. Assume that I_x' has been defined for every x, which precedes y (write xWy). Since

$$\Big| \bigcup_{xWy} I_x' \Big| \leqslant \mathfrak{n}\mathfrak{r} = \mathfrak{r} < |I_y - J|,$$

by (16), there exists $I_y' \subseteq I_y - J$ such that $| I_y' | = \mathfrak{r}$ and $I_y' \cap I_x' = \varnothing$ for xWy. The definition of I_x' for $x \in M$ is complete by transfinite induction.

For each $x \in M$ there exists a 1-1 mapping $v \to \alpha(x, v)$ from ω (with $| \omega | = \mathfrak{r}$) to I_x'. Since $x \in M_{\alpha(x,v)} \subseteq M$, it follows that

$$M = \bigcup_{x \in M} M_{\alpha(x,v)}, \qquad \text{for} \quad v \in \omega,$$

and (B) follows since the indices $\alpha(x, v)$ are distinct.

Remark. If the sets M_i, $i \in I$, and $\mathfrak{r}$ are at most enumerable, then one can prove (2) with an enumerable number of sets in the equalities. This follows by the recursion used in the proof of (A) repeated an enumerable number of steps. The system of representatives, with $\mathfrak{r}$ repetitions of each element we need (see equation 1), is given at the end of proof of (B).

If the sets M_i are finite then (2) is true with all $| I_v | = 1$, even when $| \omega | = | I |$, if $| I |$ is regular.

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References

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